Int. J. Heat Mass Transfer. Vol. 32, No. 3, pp. 611–613, 1989
Printed in Great Britain

0017–9310/89 \$3.00+0.00

TECHNICAL NOTES

An extension of the Kantorovich method and its application to a steady state heat conduction problem

PATRICIO A. A. LAURA
Institute of Applied Mechanics (CONICET), 8111 Puerto Belgrano Naval Base, Argentina

and

VICTOR H. CORTINEZ
Department of Engineering, Universidad Nacional del Sur, 8000 Bahía Blanca, Argentina

(*Received* 4 *January* 1988 *and in final form* 19 *May* 1988)

INTRODUCTION

AS POINTED out in an excellent paper [1] "approximate analytical solutions to partial differential equations are useful when exact analytical solutions are either too difficult or impossible to obtain, or when the work to find a numerical solution cannot be justified".†

Among the available analytical, approximate techniques the Kantorovich method must be cited. This method has the particular advantage over the Ritz or the Galerkin method in that one specifies completely the functions, say $f_n(x)$, in the x-direction but leaves the function in the y-direction, say $g(y)$, unspecified and to be determined by variational calculus [3]. The use of this approximation in variational formulations leads to a set of ordinary differential equations in terms of $g_n(y)$ to be considered with proper boundary conditions in the y-direction.

An extension of the Kantorovich method is hereby proposed by introducing two optimization parameters, say 'γ' and 'ξ', in $f_n(x)$, in such a manner that further optimization of the desired solution is made possible by minimizing the governing functional with respect to γ and ξ.

The idea of including an undefined parameter as an exponent included in the coordinate function was originally suggested by Lord Rayleigh [4] and extended to the determination of optimized, higher eigenvalues by Laura and Cortinez [5].

EXTENSION OF THE KANTOROVICH METHOD

Consider the case of a rectangular bidimensional domain when heat generation takes place under a steady-state situation and subject to homogeneous boundary conditions. The problem is then governed by the differential system

$$\nabla^2\theta(x,y)+\frac{p(x,y)}{k}=0 \tag{1}$$

$$\theta[L(x,y)=0]=0 \tag{2}$$

where $L(x,y)=0$ is the functional relation which defines the boundary of the domain.

Solving equation (1) is equivalent to setting up the variational formulation

$$\delta I=\int_{-b}^{b}\int_{-a}^{a}\left(\nabla^2\theta+\frac{p}{k}\right)\delta\theta\,\mathrm{d}x\,\mathrm{d}y$$
$$=\delta\left\{\int_{-b}^{b}\int_{-a}^{a}\left\{-\frac{1}{2}\left[\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2\right]+\frac{p\theta}{k}\right\}\mathrm{d}x\,\mathrm{d}y\right\} \tag{3}$$

where θ must satisfy boundary condition (2).

In accordance with the Kantorovich method one makes

$$\theta\cong\theta_a=f(x)g(y) \tag{4}$$

where $f(x)$ is constructed in such a manner that it satisfies the boundary conditions

$$f(-a)=f(a)=0 \tag{5}$$

while $g(y)$ will be determined at a later stage. In order to accomplish this one makes

$$\delta\theta=f(x)\delta g(y) \tag{6}$$

and substitutes in equation (3) to give

$$\int_{-b}^{b}\int_{-a}^{a}\left(f''fg+f^2g''+\frac{pf}{k}\right)\delta g\,\mathrm{d}x\,\mathrm{d}y. \tag{7}$$

Performing the required integration with respect to x and since

$$\int_{-a}^{a}f''f\,\mathrm{d}x=f'f\Big|_{-a}^{a}-\int_{-a}^{a}f'^2\,\mathrm{d}x=-\int_{-a}^{a}f'^2\,\mathrm{d}x \tag{8}$$

one finally obtains

$$\int_{-b}^{b}\left[-\int_{-a}^{a}f'^2\,\mathrm{d}x\cdot g+\int_{-a}^{a}f^2\,\mathrm{d}x\cdot g''+\int_{-a}^{a}\frac{pf}{k}\mathrm{d}x\right]\delta g\,\mathrm{d}y=0 \tag{9}$$

and since equation (9) must be satisfied for any variation δg one finally obtains

$$g''-gM+\frac{p_0N}{k}=0 \tag{10}$$

where

$$M=\int_{-a}^{a}f'^2\,\mathrm{d}x\Big/\int_{-a}^{a}f^2\,\mathrm{d}x \tag{11a}$$

$$N=\int_{-a}^{a}\alpha(x,y)f\,\mathrm{d}x\Big/\int_{-a}^{a}f^2\,\mathrm{d}x \tag{11b}$$

$$p(x,y)=p_0\alpha(x,y). \tag{11c}$$

Once $g(y)$ is obtained, solving equation (10) and applying the boundary conditions

† For a thorough treatment of the subject see Finlayson and Scriven's classical paper [2].

$$g(-b) = g(b) = 0 \tag{12}$$

the approximate solution of the problem is known.

However, if $f(x)$ contains two convenient optimization parameters 'γ' and 'ξ' one has

$$f = f(x, \gamma, \xi) \tag{13}$$

and the parameters M and N will also be functions of γ and ξ.

Accordingly

$$M = M(\gamma, \xi) \tag{14a}$$

$$N = N(\gamma, \xi) \tag{14b}$$

and obviously

$$g = g(y, \gamma, \xi). \tag{15}$$

Substituting now expressions (13) and (15) in equation (4) results in

$$\theta \cong \theta_a = f(x, \gamma, \xi) g(y, \gamma, \xi) \tag{16}$$

and since the governing functional

$$I = \int_{-a}^{a} \int_{-b}^{b} \left\{ -\frac{1}{2}\left[\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2\right] + \frac{p\theta}{k} \right\} \mathrm{d}x\,\mathrm{d}y \tag{17}$$

must be also a minimum with respect to γ and ξ, by requiring

$$\frac{\partial I}{\partial \gamma} = \frac{\partial I}{\partial \xi} = 0 \tag{18}$$

one is able to minimize $I[\theta]$ and as it will be shown in the following section, the solution of the field problem is optimized considerably.

APPLICATIONS

Consider first the case where

$$p(x, y) = p_0. \tag{19}$$

Since in this case the problem is symmetric with respect to the coordinate axes shown in Fig. 1 it is convenient to select $f(x)$ as an even function and for the particular case under study one chooses

$$f(x) = \left[1 - \left(\frac{x}{a}\right)^{\gamma}\right] + \xi\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^{\gamma+2}\right]; \quad x \geqslant 0 \tag{20a}$$

$$f(x) = \left[1 - \left|\frac{x}{a}\right|^{\gamma}\right] + \xi\left[\left|\frac{x}{a}\right|^2 - \left|\frac{x}{a}\right|^{\gamma+2}\right]; \quad x < 0. \tag{20b}$$

Substituting equation (20a) in equations (11a) and (11b) then in equation (10) one obtains

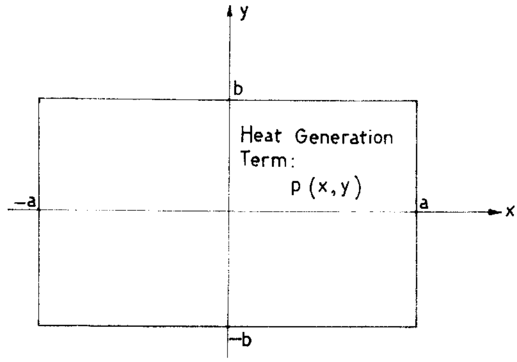

FIG. 1. Rectangular domain.

$$g(y, \gamma, \xi) = \frac{N}{M}\frac{p_0}{k}\left(1 - \frac{\cosh\sqrt{(Ma^2)}\,\frac{y}{a}}{\cosh\sqrt{(Ma^2)}\,\frac{b}{a}}\right) \tag{21}$$

and the approximate solution of the problem is then given by

$$\frac{\theta}{p_0 a^2/k} \cong \left\{\left[1 - \left(\frac{x}{a}\right)^{\gamma}\right] + \xi\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^{\gamma+2}\right]\right\} \times \frac{N}{Ma^2}\left(1 - \frac{\cosh\sqrt{(Ma^2)}\,\frac{y}{a}}{\cosh\sqrt{(Ma^2)}\,\frac{b}{a}}\right), \quad 0 \leqslant x \leqslant a/2 \tag{22}$$

and the same expression being valid for $-a/2 \leqslant x \leqslant 0$ but one must take the absolute value of x.

Table 1 depicts a comparison of maximum values of $\theta/(p_0 a^2/k)$. One observes immediately the fact that for $b/a = 1$ and 2 the approach presented in this paper yields results which, from a practical viewpoint, agree with the

Table 1. Values of $\theta/(p_0 a^2/k)$ at $x/a = y/b = 0$ for the case where $p(x, y) = p_0$

b/a	Classical Kantorovich method [3]	Exact results [3]	Present study	γ	ξ
1	0.30259	0.29469	0.29479	1.9	0.25
2	0.45774	0.45687	0.45682	1.9	0.1
4	0.49821	0.49807	0.49821	2	0

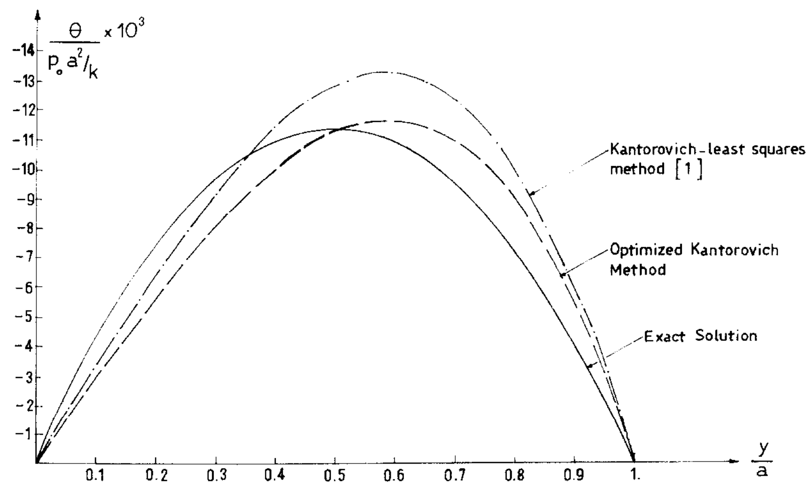

FIG. 2. Comparison of solutions for the case where $p(x, y) = -0.2p_0(y/a)$ and for $x/a = 0$ (square domain).

exact values. For $b/a = 4$ the classical and the optimized Kantorovich method yield the same value of the dimensionless temperature.

Consider now the case where

$$p(x,y) = -0.2p_0\frac{y}{b} \quad (23)$$

which is the problem studied by Djukic and Atanackovic [1] and where their interesting extension of the Kantorovich method is tested. The problem is symmetric with respect to the x-variable and antisymmetric with respect to y. Accordingly the previously obtained expressions for $f(x, \gamma, \xi)$, M and N are applicable for this situation.

On the other hand the determination of $g(y)$ is straightforward.

In the case of a square shape the maximum value of $|\theta/(p_0a^2/k)|$ is 0.01138 while the approach presented in ref. [1] yields 0.01324. The present, optimized Kantorovich method yields 0.01165 which is in excellent agreement with the exact result.

Figure 2 yields a comparison of dimensionless temperature values as a function of y/a for $x/a = 0$ in the case of a square shape. The values of γ and ξ which minimize the functional are 6.7 and -0.5, respectively.

Acknowledgements—The present investigation has been sponsored by CONICET Research and Development Program (PID 3009400).

REFERENCES

1. D. S. Djukic and T. M. Atanackovic , The least squares method: Kantorovich approach, *Int. J. Heat Mass Transfer* **24**, 443–448 (1981).
2. B. A. Finlayson and L. E. Scriven, The method of weighted residuals and its relation to certain variational principles for the analysis of transport processes, *Chem. Engng Sci.* **20**, 395–404 (1965).
3. V. Arpaci, *Conduction Heat Transfer*. Addison-Wesley, Reading, Massachusetts (1966).
4. J. W. S. Rayleigh, *Theory of Sound*, 2nd Edn, Vol. I. London (1894); reprinted by Dover, New York (1945).
5. P. A. A. Laura and V. H. Cortinez, Optimization of eigenvalues when using the Galerkin method, *A.I.Ch.E. Jl* **32**, 1025–1026 (1987).

Int. J. Heat Mass Transfer. Vol. 32, No. 3, pp. 613–616, 1989
Printed in Great Britain
0017–9310/89 $3.00+0.00

Two temperature, two phase heat transfer in porous media: solution to linear models

JAMES BAKER-JARVIS
Department of Physics, North Dakota State University, Fargo, ND 58105, U.S.A.

(*Received* 30 *October* 1987 *and in final form* 27 *May* 1988)

1. INTRODUCTION

TWO TEMPERATURE heat transfer in porous media and packed beds can be described by two coupled partial differential equations for the solid and fluid phases. An analytical solution for two phase heat transfer which includes time dependence, and heat source terms is important for modeling in such applications as predicting wall heat loss in reactors. Exact solutions to the coupled system are useful in that, in some cases, they require much less programming and computer time than finite difference solutions. Past solution techniques have proceeded by neglecting various terms in the equations [1, 2]. Burch *et al.* [3] introduced the idea of the interacting boundary conditions at the inlet face of the reactor for the two phase problem. Toovey and Dayan [4] presented an ingenious solution when fluid diffusion and heat capacity are neglected. What one feels is needed is a very general treatment of the heat transfer problem in porous media. In particular one contends that for problems with low Reynolds numbers it is not valid to neglect the diffusivity of the fluid or heat capacity (particularly when the fluid is very dense). In this note two new solutions to the time dependent, non-homogeneous problem are developed. The first solution utilizes an eigenfunction expansion and will be useful for problems when diffusion and convection are of the same order of magnitude. The solution is based on an eigenfunction expansion in the axial coordinate. Numerical results are presented for the case of a cylindrical reactor. The second solution is exact with no approximations and is valid for problems where source terms do not depend on position in the sample. A uniqueness proof has been developed and is presented elsewhere [2].

2. THE PROBLEM

One wishes to determine the distribution of temperatures of solid and fluid phases in a packed bed where the fluid may be moving relative to the solid. Let the temperature of the fluid and solid be T_f and T_s. The superficial velocity of the fluid is v_f. The differential equations for the heat transfer are

$$\frac{\partial T_f}{\partial t} + \bar{v}_f \cdot \nabla T_f = \nabla^2 T_f - h_f(T_f - T_s) + g_f(r, t)$$

$$\frac{\partial T_s}{\partial t} = \alpha' \nabla^2 T_s + h_s(T_f - T_s) + g_s(r, t) \quad (1)$$

h' is the dimensional heat transfer coefficient, $h_f = h'L^2/k_f$, $h_s = h'L^2\rho_f c_f/\rho_s c_s k_f$; α_f, α_s are thermal diffusivities; g_f and g_s are the non-dimensional source terms for fluid and solid, respectively. Also the non-dimensional variables t, z, T, r, v are related to dimensional quantities t', z', r', T', v' by $t' = t\tau_0$, $z' = zL$, $r' = rL$, $T' = T'_0 T$, $v' = v_f L/\tau_0$ and $\tau_0 = L^2/\alpha_f$, $\alpha' = \alpha_s/\alpha_f$. The initial conditions are given by

$$T_f(t = 0) = F_f(\bar{r}), \quad T_s(t = 0) = F_s(\bar{r}). \quad (2)$$

The boundary conditions on the bounding surfaces are determined by an analysis of the problem of uniqueness [2]. In general one can write

$$\nabla T_v \cdot \bar{n} = H(T_v - T_0) \quad (3)$$

where (v = s, f), H is a constant and T_0 is the temperature of the surroundings.